UNIQUENESS OF THE FISHER–RAO METRIC ON THE SPACE OF SMOOTH DENSITIES

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Abstract. On a closed manifold of dimension greater than one, every smooth weak Riemannian metric on the space of smooth positive probability densities, that is invariant under the action of the diffeomorphism group, is a multiple of the Fisher–Rao metric.

Introduction. The Fisher–Rao metric on the space $\mathrm{Prob}(M)$ of probability densities is of importance in the field of information geometry. Restricted to finite-dimensional submanifolds of $\mathrm{Prob}(M)$, so-called statistical manifolds, it is called Fisher's information metric [1]. The Fisher–Rao metric has the property that it is invariant under the action of the diffeomorphism group. The interesting question is whether it is the unique metric possessing this invariance property. A uniqueness result was established [4, p. 156] for Fisher's information metric on finite sample spaces and [2] extended it to infinite sample spaces.

The Fisher–Rao metric on the infinite-dimensional manifold of all positive probability densities was studied in [5], including the computation of its curvature. A consequence of our main theorem in this article is the infinite-dimensional analogue of the result in [4]:

Theorem. *Let M be a compact manifold without boundary of dimension ≥ 2. Then any smooth weak Riemannian metric on the space* $\mathrm{Prob}(M)$ *of smooth positive probability densities, that is invariant under the action of the diffeomorphism group of M, is a multiple of the Fisher–Rao metric.*

The situation for a 1-dimensional manifold is described at the end of the paper. Our result holds for smooth positive probability densities on a compact manifold. However, the proof can be adapted to a suitable (and there are many choices) space of densities on a non-compact manifold. In [2] the authors prove a related result about the uniqueness of an invariant 2-tensor field on the space of probability densities. However they assume that the tensor is defined also on non-smooth densities and is invariant not only under smooth diffeomorphisms, but under all sufficient statistics. This is a stronger invariance assumption, allowing the authors

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to consider probability densities that are step functions, thus reducing the problem to the finite-dimensional case of [4].

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The space of densities. Let M^m be a smooth manifold without boundary. Let (U_α, u_α) be a smooth atlas for it. The *volume bundle* $(\mathrm{Vol}(M), \pi_M, M)$ of M is the 1-dimensional vector bundle (line bundle) which is given by the following cocycle of transition functions:

$$\psi_{\alpha\beta} : U_{\alpha\beta} = U_\alpha \cap U_\beta \to \mathbb{R} \setminus \{0\} = GL(1, \mathbb{R}),$$
$$\psi_{\alpha\beta}(x) = |\det d(u_\beta \circ u_\alpha^{-1})(u_\alpha(x))| = \frac{1}{|\det d(u_\alpha \circ u_\beta^{-1})(u_\beta(x))|}.$$

Vol(M) is a trivial line bundle over M. But there is no natural trivialization. There is a natural order on each fiber. Since $\mathrm{Vol}(M)$ is a natural bundle of order 1 on M, there is a natural action of the group $\mathrm{Diff}(M)$ on $\mathrm{Vol}(M)$, given by

$$\begin{tikzcd} \mathrm{Vol(M)} \arrow[r, "|\det(T\varphi^{-1})|\circ\varphi"] \arrow[d] & \mathrm{Vol(M)} \arrow[d] \\ M \arrow[r, "\varphi"] & M \end{tikzcd} .$$

If M is orientable, then $\mathrm{Vol} = \Lambda^m T^* M$. If M is not orientable, let $\tilde{M}$ be the orientable double cover of M with its deck-transformation $\tau : \tilde{M} \to \tilde{M}$. Then $\Gamma(\mathrm{Vol}(M))$ is isomorphic to the space $\{\omega \in \Omega^m(\tilde{M}) : \tau^*\omega = -\omega\}$. These are the 'formes impaires' of de Rham. See [10, 13.1] for this.

Sections of the line bundle $\mathrm{Vol}(M)$ are called densities. The space $\Gamma(\mathrm{Vol}(M))$ of all smooth sections is a Fréchet space in its natural topology; see [9]. For each section α of $\mathrm{Vol}(M)$ of compact support the integral $\int_M \alpha$ is invariantly defined as follows: Let (U_α, u_α) be an atlas on M with associated trivialization $\psi_\alpha : \mathrm{Vol}(M)|_{U_\alpha} \to \mathbb{R}$, and let f_α be a partition of unity with $\mathrm{supp}(f_\alpha) \subset U_\alpha$. Then we put

$$\int_M \mu = \sum_\alpha \int_{U_\alpha} f_\alpha \mu := \sum_\alpha \int_{u_\alpha(U_\alpha)} f_\alpha(u_\alpha^{-1}(y)).\psi_\alpha(\mu(u_\alpha^{-1}(y)))\, dy.$$

The integral is independent of the choice of the atlas and the partition of unity.

The Fisher–Rao metric. Let M^m be a smooth compact manifold without boundary. We denote by $\mathrm{Dens}_+(M)$ the space of smooth positive densities on M, i.e. $\mathrm{Dens}_+(M) = \{\mu \in \Gamma(\mathrm{Vol}(M)) : \mu(x) > 0\ \forall x \in M\}$. Let $\mathrm{Prob}(M)$ be the subspace of positive densities with integral 1 on M. Both spaces are smooth Fréchet manifolds, in particular they are open subsets of the affine spaces of all densities and densities of integral 1 respectively. For $\mu \in \mathrm{Dens}_+(M)$ we have $T_\mu \mathrm{Dens}_+(M) = \Gamma(\mathrm{Vol}(M))$ and for $\mu \in \mathrm{Prob}(M)$ we have

$$T_\mu \mathrm{Prob}(M) = \{\alpha \in \Gamma(\mathrm{Vol}(M)) : \int_M \alpha = 0\}.$$

The Fisher–Rao metric is a Riemannian metric on $\mathrm{Prob}(M)$ and is defined as follows:

$$G^{\mathrm{FR}}_\mu(\alpha,\beta) = \int_M \frac{\alpha}{\mu}\frac{\beta}{\mu}\mu.$$

This metric is invariant under the associated action of $\mathrm{Diff}(M)$ on $\mathrm{Prob}(M)$, since

$$\Big((\varphi^*)^* G^{\mathrm{FR}}\Big)_\mu(\alpha,\beta) = G^{\mathrm{FR}}_{\varphi^*\mu}(\varphi^*\alpha,\varphi^*\beta) = \int_M \Big(\frac{\alpha}{\mu}\circ\varphi\Big)\Big(\frac{\beta}{\mu}\circ\varphi\Big)\varphi^*\mu = \int_M \frac{\alpha}{\mu}\frac{\beta}{\mu}\mu\,.$$

The uniqueness result for the Fisher–Rao metric follows from the following classification of $\mathrm{Diff}(M)$-invariant bilinear forms on $\mathrm{Dens}_+(M)$.

Main Theorem. *Let M be a compact manifold without boundary of dimension ≥ 2. Let G be a smooth (equivalently, bounded) bilinear form on* $\mathrm{Dens}_+(M)$ *which is invariant under the action of* $\mathrm{Diff}(M)$. *Then*

$$G_\mu(\alpha,\beta) = C_1 \int_M \frac{\alpha}{\mu}\frac{\beta}{\mu}\,\mu + C_2 \int_M \alpha \cdot \int_M \beta$$

for some constants C_1, C_2.

To see that this theorem implies the uniqueness of the Fisher–Rao metric, note that if G is a $\mathrm{Diff}(M)$-invariant Riemannian metric on $\mathrm{Prob}(M)$, then we can equivariantly extend it to $\mathrm{Dens}_+(M)$ via

$$G_\mu(\alpha,\beta) = G_{\mu(M)^{-1}\mu}\left(\alpha - \mu(M)\int_M \alpha, \beta - \mu(M)\int_M \beta\right)\,.$$

Relations to right-invariant metrics on diffeomorphism groups. Let $\mu_0 \in \mathrm{Prob}(M)$ be a fixed smooth positive probability density. In [7] it has been shown, that the degenerate, $\dot H^1$-metric $\frac{1}{2}\int_M \mathrm{div}^{\mu_0}(X).\,\mathrm{div}^{\mu_0}(X).\mu_0$ on $\mathfrak{X}(M)$ is invariant under the adjoint action of $\mathrm{Diff}(M,\mu_0)$. Thus the induced degenerate right invariant metric on $\mathrm{Diff}(M)$ descends to a metric on

$$\mathrm{Prob}(M) \cong \mathrm{Diff}(M,\mu_0)\backslash\,\mathrm{Diff}(M) \qquad \text{via} \qquad \mathrm{Diff}(M) \ni \varphi \mapsto \varphi^*\mu_0 \in \mathrm{Prob}(M)$$

which is invariant under the right action of $\mathrm{Diff}(M)$. This metric turns out to be the Fisher–Rao metric on $\mathrm{Prob}(M)$. In [11], the $\dot H^1$-metric was extended to a non-degenerate metric on $\mathrm{Diff}(M)$, that also descends to the Fisher–Rao metric. A consequence of our uniqueness result is the following:

Corollary. *Let* $\dim(M) \geq 2$. *If a weak right-invariant (possibly degenerate) Riemannian metric $\tilde G$ on* $\mathrm{Diff}(M)$ *descends to a metric G on* $\mathrm{Prob}(M)$*, i.e., the map* $(\mathrm{Diff}(M),\tilde G) \to (\mathrm{Prob}(M), G)$ *is a Riemannian submersion, then G has to be a multiple of the Fisher–Rao metric.*

For $M = S^1$ the descending property is much less restrictive, since in this case the group of volume preserving diffeomorphism is generated by constant vector fields only. Thus any right invariant metric on the homogenous space $\mathrm{Diff}(S^1)/S^1$ descends to a $\mathrm{Diff}(S^1)$ invariant metric on $\mathrm{Prob}(S^1)$, e.g., the homogenous Sobolev metric of order $n \geq 1$:

$$G_{\mathrm{Id}}(X,Y) = \sum_{k=1}^n \int_{S^1} \partial_\theta^k X.\partial_\theta^k Y\, d\theta\,.$$

For $n = 1$ the metric descends to the Fisher–Rao metric and for $n = 2$ we obtain a higher order metric. For the one-dimension situation see also the last Section of this article, where relations between metrics on $\operatorname{Dens}_+(S^1)$ and $\operatorname{Met}(S^1)$ are discussed.

Proof of the Main Theorem. Let us first reduce the case of a non-orientable manifold to orientable manifolds. If M is non-orientable, let $\tilde M$ be the orientable double cover and $\tau : \tilde M \to \tilde M$ the deck-transformation. We can decompose

$$\Omega^m(\tilde M) = \{\tau^*\omega = -\omega\} \oplus \{\tau^*\omega = \omega\}\,,$$

and $\operatorname{Dens}_+(M)$ is isomorphic to the first summand. Any bilinear form G on $\operatorname{Dens}_+(M)$ can be extended to a bilinear form $\tilde G$ on $\operatorname{Dens}_+(\tilde M)$ and the extension is $\operatorname{Diff}(\tilde M)$-invariant. Thus we have reduced the proof to the orientable situation.

From now on we assume that M is orientable. Let us fix a basic probability density μ_0. By the Moser trick [12], see [10, 31.13] or the proof of [9, 43.7] for proofs in the notation used here, there exists for each $\mu \in \operatorname{Dens}_+(M)$ a diffeomorphism $\varphi_\mu \in \operatorname{Diff}(M)$ with $\varphi_\mu^*\mu = \mu(M)\mu_0 =: c.\mu_0$ where $c = \mu(M) = \int_M \mu > 0$. Then

$$\big((\varphi_\mu^*)^*G\big)_\mu(\alpha,\beta) = G_{\varphi_\mu^*\mu}(\varphi_\mu^*\alpha,\varphi_\mu^*\beta) = G_{c.\mu_0}(\varphi_\mu^*\alpha,\varphi_\mu^*\beta)\,.$$

Thus it suffices to show that for any $c > 0$ we have

$$G_{c\mu_0}(\alpha,\beta) = \frac{c_1}{c}.\int_M \frac{\alpha}{\mu_0}\frac{\beta}{\mu_0}\mu_0 + C_2\int_M \alpha\cdot\int_M \beta$$

for some constants c_1, C_2. Both bilinear forms are still invariant under the action of the group $\operatorname{Diff}(M, c\mu_0) = \operatorname{Diff}(M,\mu_0) = \{\psi \in \operatorname{Diff}(M) : \psi^*\mu_0 = \mu_0\}$. The bilinear form

$$T_{\mu_0}\operatorname{Dens}_+(M) \times T_{\mu_0}(M)\operatorname{Dens}_+ \ni (\alpha,\beta) \mapsto G_{c\mu_0}\Big(\frac{\alpha}{\mu_0}\mu_0, \frac{\beta}{\mu_0}\mu_0\Big)$$

can be viewed as a bilinear form

$$C^\infty(M) \times C^\infty(M) \ni (f,g) \mapsto G_c(f,g)\,.$$

We will consider now the associated bounded mapping

$$\check G_c : C^\infty(M) \to C^\infty(M)' = \mathcal D'(M)\,.$$

(1) Since we assume that M is orientable, each density is an m-form. The Lie algebra $\mathfrak X(M,\mu_0)$ of $\operatorname{Diff}(M,\mu_0)$ consists of vector fields X with $\operatorname{div}^{\mu_0}(X) = 0$, or $di_X\mu_0 = 0$. The mapping $\hat\imath_{\mu_0} : \mathfrak X(M) \to \Omega^{m-1}(M)$ given by $X \mapsto i_X\mu_0$ is an isomorphism. The Lie subalgebra $\mathfrak X(M,\mu_0)$ of divergence free vector fields corresponds to the space of closed $(m-1)$-forms. Denote by $\mathfrak X_{\text{exact}}(M,\mu_0)$ the space of 'exact' divergence free vector fields $X = \hat\imath_{\mu_0}^{-1}(d\omega)$, where ω runs through $\Omega^{m-2}(M)$.

(2) *If for $f \in C^\infty(M)$ and a connected open set $U \subseteq M$ we have $\mathcal L_X(f|U) = 0$ for all $X \in \mathfrak X_{exact}(U,\mu_0)$, then $f|U$ is constant.*

Since we shall need some details later on, we prove this well-known fact. Let $x \in U$. For every tangent vector $X_x \in T_xM$ we can find a vector field $X \in \mathfrak X_{\text{exact}}(M,\mu_0)$ such that $X(x) = X_x$; to see this, choose a chart (U,u) near x such that $\mu_0|U = du^1\wedge\cdots\wedge du^m$, and choose $g \in C_c^\infty(U)$, such that $g = 1$ near x. Then $X := \hat\imath_{\mu_0}^{-1} d(g.u^2.du^3\wedge\cdots\wedge du^m) \in \mathfrak X_{\text{exact}}(M,\mu_0)$ and $X = \partial_{u^1}$ near x. So we can produce a basis for T_xM and even a local frame near x. Thus $\mathcal{L}_X f|U = 0$ for all $X \in \mathfrak{X}_{\mathrm{exact}}(M,\mu_0)$ implies $df = 0$ and hence f is constant.

(3) *If for a distribution $A \in \mathcal{D}'(M)$ and a connected open set $U \subseteq M$ we have $\mathcal{L}_X A|U = 0$ for all $X \in \mathfrak{X}_{exact}(M,\mu_0)$, then $A|U = C\mu_0|U$ for some constant C, meaning $\langle A, f\rangle = C\int_M f\mu_0$ for all $f \in C_c^\infty(U)$.*

Because $\langle \mathcal{L}_X A, f\rangle = -\langle A, \mathcal{L}_X f\rangle$, the invariance property, $\mathcal{L}_X A|U = 0$, implies $\langle A, \mathcal{L}_X f\rangle = 0$ for all $f \in C_c^\infty(U)$. Clearly, $\int_M (\mathcal{L}_X f)\mu_0 = 0$. Without loss, let us assume now that U is an open chart, that is diffeomorphic to $\mathbb{R}^m$. Let $g \in C_c^\infty(U)$ satisfy $\int_M g\mu_0 = 0$; we will show that $\langle A, g\rangle = 0$. Because the integral over $g\mu_0$ is zero, the compact cohomology class $[g\mu_0] \in H_c^m(U) \cong \mathbb{R}$ vanishes; thus there exists $\alpha \in \Omega_c^{m-1}(U) \subset \Omega^{m-1}(M)$ with $d\alpha = g\mu_0$. Since we are working on a coordinate chart, which is diffeomorphic to $\mathbb{R}^m$, we can write $\alpha = \sum_j f_j d\beta_j$ with $\beta_j \in \Omega^{m-2}(U)$ and $f_j \in C_c^\infty(U)$. Choose $h \in C_c^\infty(U)$ with $h = 1$ on $\bigcup_j \operatorname{supp}(f_j)$, so that $\alpha = \sum_j f_j d(h\beta_j)$ and $h\beta_j \in \Omega^{m-2}(M)$. In particular the vector fields $X_j = \hat{\iota}_{\mu_0}^{-1} d(h\beta_j)$ lie in $\mathfrak{X}_{\mathrm{exact}}(M,\mu_0)$ and we have the identity $\sum_j f_j . i_{X_i}\mu_0 = \alpha$. This means

$$\sum_j (\mathcal{L}_{X_j} f_j)\mu_0 = \sum_j \mathcal{L}_{X_j}(f_j\mu_0) = \sum_j d i_{X_j}(f_j\mu_0) = d\Big(\sum_j f_j . i_{X_j}\mu_0\Big) = d\alpha = g\mu_0$$

$$\sum_j \mathcal{L}_{X_j} f_j = g\,,$$

leading to

$$\langle A, g\rangle = \sum_j \langle A, \mathcal{L}_{X_j} f_j\rangle = -\sum_j \langle \mathcal{L}_{X_j} A, f_j\rangle = 0\,.$$

So $\langle A, g\rangle = 0$ for all $g \in C_c^\infty(U)$ with $\int_M g\mu_0 = 0$. Finally, choose a function φ with support in U and $\int_M \varphi\mu_0 = 1$. Then for any $f \in C_c^\infty(U)$, the function defined by $g = f - (\int_M f\mu_0).\varphi$ in $C^\infty(M)$ satisfies $\int_M g\mu_0 = 0$ and so

$$\langle A, f\rangle = \langle A, g\rangle + \langle A, \varphi\rangle \int_M f\mu_0 = C\int_M f\mu_0\,,$$

with $C = \langle A, \varphi\rangle$. Thus $A|U = C\mu_0|U$ and (3) is proved.

(4) *The operator $\check{G}_c : C^\infty(M) \to \mathcal{D}'(M)$ has the following property: If for $f \in C^\infty(M)$ and a connected open $U \subseteq M$ the restriction $f|U$ is constant, then we have $\check{G}(f)|U = C_U(f)\mu_0|U$ for some constant $C_U(f)$.*

To see (4), for $x \in U$, choose a smooth function g on M with $g = 1$ in a neighborhood of $M \setminus U$ and $g = 0$ on an open neighborhood V of x. Then for any $X \in \mathfrak{X}_{\mathrm{exact}}(M,\mu_0)$, that is $X = \hat{\iota}_{\mu_0}^{-1}(d\omega)$ for some $\omega \in \Omega^{m-2}(M)$, let $Y = \hat{\iota}_{\mu_0}^{-1}(d(g\omega))$. The vector field Y is again divergence free, equals X on a neighborhood of $M \setminus U$, and vanishes on V. Since f is constant on U, it follows that $\mathcal{L}_X f = \mathcal{L}_Y f$. Using the invariance of G_c, we have for all $h \in C^\infty(M)$,

$$\big\langle \mathcal{L}_X \check{G}_c(f), h\big\rangle = \big\langle \check{G}_c(f), -\mathcal{L}_X h\big\rangle = -G_c(f, \mathcal{L}_X h) = G_c(\mathcal{L}_X f, h) = \big\langle \check{G}_c(\mathcal{L}_X f), h\big\rangle\,,$$

and thus also

$$\mathcal{L}_X \check{G}_c(f) = \check{G}_c(\mathcal{L}_X f) = \check{G}_c(\mathcal{L}_Y f) = \mathcal{L}_Y \check{G}_c(f)\,.$$

Now Y vanishes on V and therefore so does $\mathcal{L}_X \check{G}_c(f)$. By (3) we have $\check{G}_c(f)|V = C_V(f)\mu_0|V$ for some $C_V(f) \in \mathbb{R}$. Since U is connected, all the constants $C_V(f)$ have to agree, giving a constant $C_U(f)$, depending only on U and f. Thus (4) follows.

By the Schwartz kernel theorem, $\check{G}_c$ has a kernel $\hat{G}_c$, which is a distribution (generalized function) in

$$\mathcal{D}'(M \times M) \cong \mathcal{D}'(M)\bar{\otimes}\mathcal{D}'(M) = (C^\infty(M)\bar{\otimes}C^\infty(M))' \cong L(C^\infty(M), \mathcal{D}'(M))\,.$$

Note the defining relations

$$G_c(f,g) = \langle \check{G}_c(f), g\rangle = \langle \hat{G}_c, f \otimes g\rangle.$$

Moreover, $\hat{G}_c$ is invariant under the diagonal action of $\operatorname{Diff}(M, \mu_0)$ on $M \times M$. In view of the tensor product in the defining relations, the infinitesimal version of this invariance is: $\mathcal{L}_{X\times 0+0\times X}\hat{G}_c = 0$ for all $X \in \mathfrak{X}(M, \mu_0)$.

(5) *There exists a constant C_2 such that the distribution $\hat{G}_c - C_2\mu_0 \otimes \mu_0$ is supported on the diagonal of $M \times M$.*

Namely, if $(x, y) \in M \times M$ is not on the diagonal, then there exist open neighborhoods U_x of x and U_y of y in M such that $\overline{U_x} \times \overline{U_y}$ is disjoint to the diagonal, or $\overline{U_x} \cap \overline{U_y} = \emptyset$. Choose any functions $f, g \in C^\infty(M)$ with $\operatorname{supp}(f) \subset U_x$ and $\operatorname{supp}(g) \subset U_y$. Then $f|(M \setminus \overline{U_x}) = 0$, so by (4), $\check{G}_c(f)|(M \setminus \overline{U_x}) = C_{M\setminus\overline{U_x}}(f).\mu_0$. Therefore,

$$\begin{aligned}
G_c(f,g) &= \langle \hat{G}_c, f \otimes g\rangle = \langle \check{G}_c(f), g\rangle \\
&= \langle \check{G}_c(f)|(M \setminus \overline{U_x}), g|(M \setminus \overline{U_x})\rangle\,, \qquad \text{since } \operatorname{supp}(g) \subset U_y \subset M \setminus \overline{U_x}, \\
&= C_{M\setminus\overline{U_x}}(f) \cdot \int_M g\mu_0
\end{aligned}$$

By applying the argument for the transposed bilinear form $G_c^T(g,f) = G_c(f,g)$, which is also $\operatorname{Diff}(M,\mu_0)$-invariant, we arrive at

$$G_c(f,g) = G_c^T(g,f) = C'_{M\setminus\overline{U_y}}(g) \cdot \int_M f\mu_0\,.$$

Fix two functions f_0, g_0 with the same properties as f, g and additionally $\int_M f_0\mu_0 = 1$ and $\int_M g_0\mu_0 = 1$. Then we get $C_{M\setminus\overline{U_x}}(f) = C'_{M\setminus\overline{U_y}}(g_0)\int_M f\mu_0$, and so

$$\begin{aligned}
G_c(f,g) &= C'_{M\setminus\overline{U_y}}(g_0)\int_M f\mu_0 \cdot \int_M g\mu_0 \\
&= C_{M\setminus\overline{U_x}}(f_0)\int_M f\mu_0 \cdot \int_M g\mu_0\,.
\end{aligned}$$

Since $\dim(M) \geq 2$ and M is connected, the complement of the diagonal in $M \times M$ is also connected, and thus the constants $C_{M\setminus\overline{U_x}}(f_0)$ and $C'_{M\setminus\overline{U_y}}(g_0)$ cannot depend on the functions f_0, g_0 or the open sets U_x and U_y as long as the latter are disjoint. Thus there exists a constant C_2 such that for all $f, g \in C^\infty(M)$ with disjoint supports we have

$$G_c(f,g) = C_2\int_M f\mu_0 \cdot \int_M g\mu_0$$

Since $C_c^\infty(U_x \times U_y) = C_c^\infty(U_x)\bar{\otimes}C_c^\infty(U_y)$, this implies claim (5).

Now we can finish the proof. We may replace $\hat{G}_c \in \mathcal{D}'(M\times M)$ by $\hat{G}_c - C_2\mu_0\otimes\mu_0$ and thus assume without loss that the constant C_2 in (5) is 0. Let (U,u) be a chart on M such that $\mu_0|U = du^1\wedge\cdots\wedge du^m$. The distribution $\hat{G}_c|U\times U\in\mathcal{D}'(U\times U)$ has support contained in the diagonal and is of finite order k. By [6, Theorem 5.2.3], the corresponding operator $\check{G}_c : C_c^\infty(U)\to\mathcal{D}'(U)$ is of the form $\hat{G}_c(f) = \sum_{|\alpha|\le k} A_\alpha.\partial^\alpha f$ for $A_\alpha\in\mathcal{D}'(U)$, so that $G(f,g) = \langle\check{G}_c(f),g\rangle = \sum_\alpha\langle A_\alpha,(\partial^\alpha f).g\rangle$. Moreover, the A_α in this representation are uniquely given, as is seen by a look at [6, Theorem 2.3.5].

For $x\in U$ choose an open set U_x with $x\in U_x\subset\overline{U_x}\subset U$, and choose $X\in\mathfrak{X}_{\text{exact}}(M,\mu_0)$ with $X|U_x = \partial_{u^i}$, as in the proof of (2). For functions $f,g\in C_c^\infty(U_x)$ we then have, by the invariance of G_c,

$$\begin{aligned}
0 &= G_c(\mathcal{L}_X f,g) + G_c(f,\mathcal{L}_X g) = \langle\hat{G}_c|U\times U, \mathcal{L}_X f\otimes g + f\otimes\mathcal{L}_X g\rangle\\
&= \sum_\alpha\langle A_\alpha, (\partial^\alpha\partial_{u^i}f).g + (\partial^\alpha f)(\partial_{u^i}g)\rangle\\
&= \sum_\alpha\langle A_\alpha, \partial_{u^i}((\partial^\alpha f).g)\rangle = \sum_\alpha\langle -\partial_{u^i}A_\alpha, (\partial^\alpha f).g\rangle\,.
\end{aligned}$$

Since the corresponding operator has again a kernel distribution which is supported on the diagonal, and since the distributions in the representation are unique, we can conclude that $\partial_{u^i}A_\alpha|U_x = 0$ for each α, and each i.

To see that this implies that $A_\alpha|U_x = C_\alpha\mu_0|U_x$, let $f\in C_c^\infty(U_x)$ with $\int_M f\mu_0 = 0$. Then, as in (3), there exists $\omega\in\Omega_c^{m-1}(U_x)$ with $d\omega = f\mu_0$. In coordinates we have $\omega = \sum_i\omega_i.du^1\wedge\cdots\wedge\widehat{du^i}\wedge du^m$, and so $f = \sum_i(-1)^{i+1}\partial_{u^i}\omega_i$ with $\omega_i\in C_c^\infty(U_x)$. Thus

$$\langle A_\alpha, f\rangle = \sum_i(-1)^{i+1}\langle A_\alpha,\partial_{u^i}\omega_i\rangle = \sum_i(-1)^i\langle\partial_{u^i}A_\alpha,\omega_i\rangle = 0\,.$$

Hence $\langle A_\alpha, f\rangle = 0$ for all $f\in C_c^\infty(U_x)$ with zero integral and as in the proof of (3) we can conclude that $A_\alpha|U_x = C_\alpha\mu_0|U_x$.

But then $G_c(f,g) = \int_{U_x}(Lf).g\mu_0$ for the differential operator $L = \sum_{|\alpha|\le k}C_\alpha\partial^\alpha$ with constant coefficients on U_x. Now we choose $g\in C_c^\infty(U_x)$ such that $g=1$ on the support of f. By the invariance of G_c we have again $0 = G_c(\mathcal{L}_X f,g) + G_c(f,\mathcal{L}_X g) = \int_{U_x}L(\mathcal{L}_X f).\mu_0$ for each $X\in\mathfrak{X}(M,\mu_0)$. Thus the distribution $f\mapsto\int_{U_x}L(f)\mu_0$ vanishes on all functions of the form $\mathcal{L}_X f$, and by (3) we conclude that $L(\quad).\mu_0 = C_x.\mu_0$ in $\mathcal{D}'(U_x)$, or $L = C_x\,\mathrm{Id}$. By covering M with open sets U_x, we see that all the constants C_x are the same. This concludes the proof of the Main Theorem. □

Invariant metrics on $\mathrm{Dens}_+(S^1)$**.** It is interesting to consider the case $M = S^1$, which is not covered by the theorem. In the following let $M = S^1$. Then positive densities can be represented by positive one-forms. The space of all positive densities is isomorphic to the space of all Riemannian metrics on S^1 via the $\mathrm{Diff}(S^1)$-equivariant mapping

$$\Phi = (\quad)^2 : \mathrm{Dens}_+(S^1)\to\mathrm{Met}(S^1),\qquad \Phi(fd\theta) = f^2d\theta^2\,.$$

On $\mathrm{Met}(S^1)$ there exists a variety of $\mathrm{Diff}(S^1)$-invariant metrics; see [3]. We can take for example the family of Sobolev-type metrics. Write $g\in\mathrm{Met}(S^1)$ in the form

$g = \tilde{g} d\theta^2$ and $h = \tilde{h} d\theta^2$, $k = \tilde{k} d\theta^2$ with $\tilde{g}, \tilde{h}, \tilde{k} \in C^\infty(S^1)$. Then for any integer n, the following metrics are $\mathrm{Diff}(S^1)$-invariant,

$$G^l_g(h,k) = \int_{S^1} \frac{\tilde{h}}{\tilde{g}}.\,(1+\Delta^g)^n \left(\frac{\tilde{k}}{\tilde{g}}\right) \sqrt{\tilde{g}}\, d\theta\,;$$

here Δ^g denotes the Laplacian on S^1 with respect to the metric g. Due to the equivariance of Φ, the pullback via Φ of any of these metrics yields a $\mathrm{Diff}(S^1)$-invariant metric on $\mathrm{Dens}_+(M)$, given by

$$G_\mu(\alpha,\beta) = 4 \int_{S^1} \frac{\alpha}{\mu}.\left(1+\Delta^{\Phi(\mu)}\right)^n \left(\frac{\beta}{\mu}\right) \mu\,.$$

For $n=0$ we obtain 4 times the Fisher–Rao metric. For $n \geq 1$ we see by the number of derivatives involved in the expression for $G_\mu(\alpha,\beta)$, that we obtain different $\mathrm{Diff}(S^1)$-invariant metrics on $\mathrm{Dens}_+(M)$ as well as on $\mathrm{Prob}(S^1)$.

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